Problem 1.40

Compute the divergence of the function

$$\mathbf{v} = (r\cos\theta)\hat{\mathbf{r}} + (r\sin\theta)\hat{\boldsymbol{\theta}} + (r\sin\theta\cos\phi)\hat{\boldsymbol{\phi}}.$$

Check the divergence theorem for this function, using as your volume the inverted hemispherical bowl of radius R, resting on the xy plane and centered at the origin (Fig. 1.40).

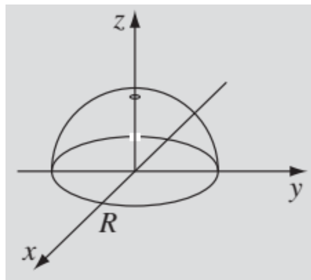

Fig. 1.40

Solution

In spherical coordinates (r, ϕ, θ), where θ is the angle from the polar axis, the divergence of a vector function is

$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}.$$

For the given function, it evaluates to

$$\begin{aligned}
\nabla\cdot\mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2(r\cos\theta)\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}[(r\sin\theta)\sin\theta] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r\sin\theta\cos\phi) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}(r^3\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r\sin^2\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r\sin\theta\cos\phi) \\
&= \frac{1}{r^2}(3r^2\cos\theta) + \frac{1}{r\sin\theta}(2r\sin\theta\cos\theta) + \frac{1}{r\sin\theta}(-r\sin\theta\sin\phi) \\
&= (3\cos\theta) + (2\cos\theta) + (-\sin\phi) \\
&= 5\cos\theta - \sin\phi.
\end{aligned}$$

The divergence theorem (or Gauss's theorem) relates the volume integral of $\nabla\cdot\mathbf{v}$ to a closed surface integral.

$$\iiint_D \nabla\cdot\mathbf{v}\,dV = \oiint_{\text{bdy } D} \mathbf{v}\cdot d\mathbf{S}$$

Here D is the inverted hemispherical bowl shown in Fig. 1.40.

Calculate the left side first.

$$\iiint\limits_D \nabla \cdot \mathbf{v}\, dV = \int_0^{\pi/2}\int_0^{2\pi}\int_0^R (5\cos\theta - \sin\phi)(r^2\sin\theta\, dr\, d\phi\, d\theta)$$

$$= \int_0^{\pi/2}\int_0^{2\pi}\int_0^R (5r^2\sin\theta\cos\theta - r^2\sin\phi\sin\theta)\, dr\, d\phi\, d\theta$$

$$= \int_0^{\pi/2}\int_0^{2\pi}\int_0^R \left(\frac{5}{2}r^2\sin 2\theta - r^2\sin\phi\sin\theta\right) dr\, d\phi\, d\theta$$

$$= \frac{5}{2}\int_0^{\pi/2}\int_0^{2\pi}\int_0^R r^2\sin 2\theta\, dr\, d\phi\, d\theta - \int_0^{\pi/2}\int_0^{2\pi}\int_0^R r^2\sin\phi\sin\theta\, dr\, d\phi\, d\theta$$

$$= \frac{5}{2}\left(\int_0^{\pi/2}\sin 2\theta\, d\theta\right)\left(\int_0^{2\pi} d\phi\right)\left(\int_0^R r^2\, dr\right) - \left(\int_0^{\pi/2}\sin\theta\, d\theta\right)\left(\int_0^{2\pi}\sin\phi\, d\phi\right)\left(\int_0^R r^2\, dr\right)$$

$$= \frac{5}{2}(1)(2\pi)\left(\frac{R^3}{3}\right) - (1)(0)\left(\frac{R^3}{3}\right)$$

$$= \frac{5\pi R^3}{3}$$

Secondly, calculate the right side. The closed surface consists of the upper hemisphere (with outward normal $\hat{\mathbf{r}}$) and the disk in the xy-plane (with outward normal $-\hat{\mathbf{z}} = \hat{\boldsymbol{\theta}}$ at $\theta = \pi/2$).

$$\oiint\limits_{\text{bdy } D} \mathbf{v}\cdot d\mathbf{S} = \int_0^{\pi/2}\int_0^{2\pi} [(r\cos\theta)\,\hat{\mathbf{r}} + (r\sin\theta)\,\hat{\boldsymbol{\theta}} + (r\sin\theta\cos\phi)\,\hat{\boldsymbol{\phi}}]\Big|_{r=R}\cdot(\hat{\mathbf{r}}R^2\sin\theta\, d\phi\, d\theta)$$

$$+ \int_0^{2\pi}\int_0^R [(r\cos\theta)\,\hat{\mathbf{r}} + (r\sin\theta)\,\hat{\boldsymbol{\theta}} + (r\sin\theta\cos\phi)\,\hat{\boldsymbol{\phi}}]\Big|_{\theta=\pi/2}\cdot(\hat{\boldsymbol{\theta}}r\, dr\, d\phi)$$

$$= \int_0^{\pi/2}\int_0^{2\pi} [(R\cos\theta)\,\hat{\mathbf{r}} + (R\sin\theta)\,\hat{\boldsymbol{\theta}} + (R\sin\theta\cos\phi)\,\hat{\boldsymbol{\phi}}]\cdot(\hat{\mathbf{r}}R^2\sin\theta\, d\phi\, d\theta)$$

$$+ \int_0^{2\pi}\int_0^R [(0)\,\hat{\mathbf{r}} + (r)\,\hat{\boldsymbol{\theta}} + (r\cos\phi)\,\hat{\boldsymbol{\phi}}]\cdot(\hat{\boldsymbol{\theta}}r\, dr\, d\phi)$$

$$= \int_0^{\pi/2}\int_0^{2\pi}(R\cos\theta)(R^2\sin\theta\, d\phi\, d\theta) + \int_0^{2\pi}\int_0^R r^2\, dr\, d\phi$$

$$= R^3\left(\int_0^{\pi/2}\sin\theta\cos\theta\, d\theta\right)\left(\int_0^{2\pi} d\phi\right) + \left(\int_0^{2\pi} d\phi\right)\left(\int_0^R r^2\, dr\right)$$

$$= R^3\left(\frac{1}{2}\right)(2\pi) + (2\pi)\left(\frac{R^3}{3}\right)$$

$$= \frac{5\pi R^3}{3}$$